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Polynomial invariants and quandles of twisted links

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Dedicated to Professor Akio Kawauchi on his 60th birthday

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ABSTRACT

Bourgoin defined the notion of a twisted link which corresponds to a stable equivalence class of links in oriented thickenings. It is a generalization of a virtual link. Some invariants of virtual links are extended for twisted links including the knot group and the Jones polynomial. In this paper, we generalize a multivariable polynomial invariant of a virtual link to a twisted link. We also introduce a quandle of a twisted link.

1. Introduction

Virtual knot theory is a generalization of knot theory which is based on Gauss chord diagrams and link diagrams on closed oriented surfaces [10]. Virtual links correspond to stable equivalence classes of links in oriented 3-manifolds which are trivial line bundles over closed oriented surfaces [3,8]. Bourgoin extended it to twisted knot theory, which is focused on link diagrams on closed, possibly non-orientable surfaces. Twisted links correspond to stable equivalence classes of links in oriented 3-manifolds which are line bundles over closed possibly non-orientable surfaces [2]. Virtual links are regarded as twisted links.

A *virtual link diagram* is a link diagram which may have *virtual crossings*, which are encircled crossings without over-under information. A *twisted link diagram* is a virtual link diagram which may have some bars on arcs. Two examples of twisted link diagrams are depicted in Fig. 1.

A *virtual link* is an equivalence class of a virtual link diagram by Reidemeister moves and virtual Reidemeister moves in Figs. 2 and 3. A *twisted link* is an equivalence class of a twisted link diagram by Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves in Figs. 2, 3 and 4.

Kauffman extended the Jones polynomial (f-polynomial), the knot group and the knot quandle to a virtual link [10]. In [2], Bourgoin extended the Jones polynomial and the knot group to a twisted link. Some invariants of classical links or virtual links are generalized to twisted links. A multivariable polynomial invariant of a virtual link is defined by Dye and Kauffman [4], and Miyazawa [12], independently. It is a refinement of the Jones polynomial. In this paper, we extend the multivariable polynomial invariant to twisted links. We also define quandles of twisted links. We give some applications of these invariants.

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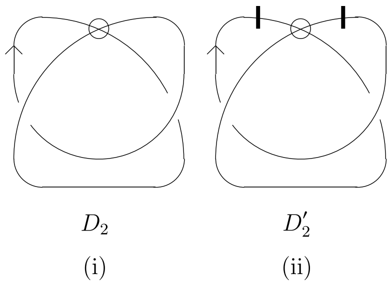


Fig. 1. Examples of twisted link diagrams.

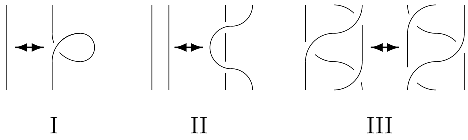


Fig. 2. Reidemeister moves.

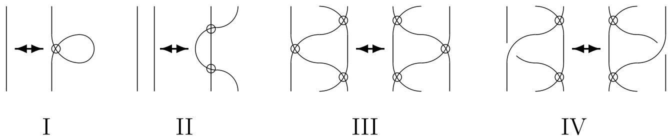


Fig. 3. Virtual Reidemeister moves.

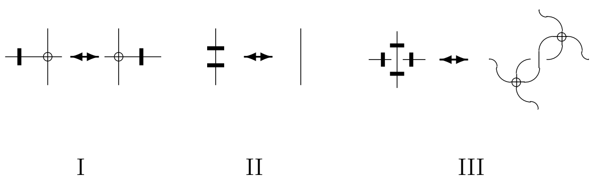


Fig. 4. Twisted Reidemeister moves.

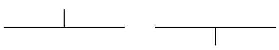


Fig. 5. Pole.

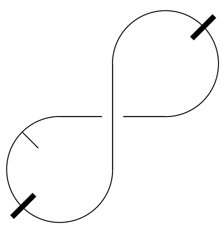


Fig. 6. An example of a pole diagram.

2. Multivariable polynomial invariants

A *pole diagram* is a twisted link diagram which may have some poles on its edges as depicted in Fig. 5 (cf. [5]). See Fig. 6 for an example of a pole diagram.

A local replacement at a real crossing of a twisted link diagram depicted in Fig. 7 indicated A or B is called *A-splice* or *B-splice*, respectively. A *state* of a twisted link diagram D is a pole diagram obtained from D by applying A-splice or B-splice at each real crossing of D. Note that a state has no real crossings.

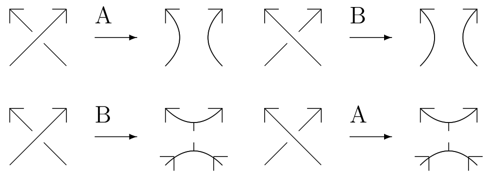
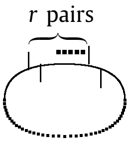
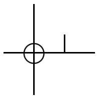
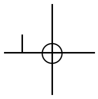


Fig. 7. Splice.

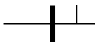
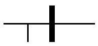
Let ℓ be a loop of a state of D. Note that the number of poles on ℓ is even, since the orientations of the edges of ℓ change alternately at each pole. We define an index, denoted by $\iota(\ell) \in \mathbb{Z}$ as follows, where we ignore orientations of all edges of ℓ.

(1) $\iota($ 

$) = r$, where $2r$ poles appear on both sides alternately, and the dotted line may have some virtual crossings and some bars.

(2) $\iota($  $) = \iota($  $)$.

(3) $\iota($  $) = \iota($ $)$.

(4) $\iota($  $) = \iota($  $)$.

Note that $\iota(\ell) = 0$ if the number of bars on ℓ is odd by (2), (3) and (4).

For a twisted link diagram D, let S be a state of D and $\omega(D)$ be the writhe of D, which is the number of the positive crossings minus that of negative ones. We denote the number of A-splices minus that of B-splices applied to obtain S from D by $\natural S$. The number of loops of S is denoted by $\sharp S$. The number of loops of S which have odd numbers of bars on them is denoted by $\sharp_o S$. The number of loops of S whose indices are i is denoted by $\tau_i(S)$.

For a state S, we define $\langle\langle D|S\rangle\rangle$ by

$$\langle\langle D \mid S\rangle\rangle = A^{\natural S}\left(-A^2 - A^{-2}\right)^{\sharp S} M^{\sharp_o S} d_1^{\tau_1(S)} d_2^{\tau_2(S)} \cdots \in \mathbb{Z}\left[A, A^{-1}, M, d_1, d_2, \ldots\right],$$

and we define $\langle\langle D\rangle\rangle$ by $\langle\langle D\rangle\rangle = \sum_S \langle\langle D|S\rangle\rangle$, where S runs over all states of D. Let $R_D = (-A^3)^{-\omega(D)}\langle\langle D\rangle\rangle$.

Theorem 1. *The polynomial R_D is an invariant of a twisted link.*

Proof. Let D and D' be twisted link diagrams such that they are related by one of Reidemeister moves, virtual Reidemeister moves and twisted Reidemeister moves. We see that R_D is equal to $R_{D'}$ by the analogous argument to that of Theorem 1 in [7]. □

By substituting 1 for d_i, R_D turns into the polynomial invariant which is equivalent to the twisted Jones polynomial $\tilde{f}_D(A, M)$ defined in [2]. By substituting $(t^i + t^{-i})/2$ for d_i, R_D turns into the polynomial invariant which is equivalent to the polynomial invariant of D defined in [7].

Proposition 2. *There is a map $\gamma : \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \ldots] \to \mathbb{Z}$ such that for any twisted link L represented by a diagram D, the real crossing number of L is equal to or greater than $\gamma(R_D)$.*

Proof. For a monomial $kA^p d_1^{\tau(1)} d_2^{\tau(2)} \cdots d_q^{\tau(q)} \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \ldots]$, we define $\kappa(kA^p d_1^{\tau(1)} d_2^{\tau(2)} \cdots d_q^{\tau(q)})$ by $\sum_{i=1}^q i\tau(i)$. The map

$$\gamma : \mathbb{Z}\left[A, A^{-1}, M, d_1, d_2, \ldots\right] \to \mathbb{Z}$$

sends $g \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \ldots]$ to the maximal of $\{\kappa(I) | I$ is a term of $g\}$. By the definition of R_D, the image of this map is less than or equal to the number of real crossings of D. □

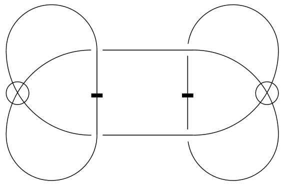


Fig. 8. An twisted link diagram.

There is a similar relationship between the virtual crossing number of a virtual link L represented by a diagram D and $\gamma(R_D)$ [4,12].

For a polynomial $g \in \mathbb{Z}[A, A^{-1}, M, d_1, d_2, \ldots]$, we denote the maximal degree (or minimal degree) of A of g by $\operatorname{Max}\deg_A g$ (or $\operatorname{Min}\deg_A g$) and $\operatorname{Span}_A g$ is defined by $\operatorname{Max}\deg_A g - \operatorname{Min}\deg_A g$.

Let D be a twisted link diagram on S^2 with μ components. Let $C = C_1 \cup \cdots \cup C_\mu$ be the disjoint union of μ circles and let $f : C \to S^2$ be an immersion such that D is the image of f with information of real crossings, virtual crossings and bars. The *Gauss diagram* means the 1-dimensional complex obtained from C by attaching some arcs corresponding to the real crossings. We call the number of the connected components of the Gauss diagram the *Gauss connected number* of D.

Theorem 3. *Let D be a twisted link diagram with m real crossings whose Gauss connected number is b. Then we have*

$$\operatorname{Span}_A(R_D) \leqq 4(m+b).$$

In particular, if D represents a non-split twisted link then $\operatorname{Span}_A(R_D) \leqq 4m+4$.

We give the proof of Theorem 3 in Section 4. A similar relation was found by S. Satoh and Y. Tomiyama for the Jones polynomial invariant and multivariable polynomial invariant for virtual links in [13].

For example, the twisted Jones polynomials of the twisted link diagram in Fig. 8 is $-A^2 - A^{-2}$. Our multivariable invariant of it is $(-A^2 - A^{-2})((A^2 + A^{-2})^2 + 2(A^2 + A^{-2}) + 1 - ((A^2 + A^{-2})^2 + 2(A^2 + A^{-2}))d_1^2)$, which means the real crossing number of it is greater than or equal to 2.

The twisted Jones polynomials of two twisted links presented by the diagrams D_2 and D_2' in Fig. 1(i) and (ii) are $-A^{-6}(A^2 + A^{-2})(A^2 + 1 - A^{-4})$. On the other hand, our multivariable polynomials of them are $-(A^2 + A^{-2})(A^{-4} + A^{-8}(A^2 - A^{-2})d_1)$ and $-A^{-6}(A^2 + A^{-2})(A^2 + 1 - A^{-4})$, respectively. We conclude that D_2 and D_2' are not equivalent. We also see that the real crossing numbers are 2 by Theorem 3.

3. Quandles of twisted links

Let D be a twisted link diagram. An *edge* of D means a connected component of $D \setminus (\{\text{real crossings}\} \cup \{\text{virtual crossings}\} \cup \{\text{bars}\})$. An edge is an arc or a simple loop. Let $e_1, \ldots, e_p$ be the edges of D. We assume that if e_i is an arc then e_{i+1} is an arc that follows e_i in the diagram unless e_i is the last edge on a component of D. The *twisted knot group* of D, denoted by $\widetilde{\Pi}(D)$, is defined as follows: The generating set is $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ where x_i and y_i $(i = 1, \ldots, p)$ are symbols associated to each edge e_i. For a positive crossing, we associate four relations given in Table 1(i), where x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j and x_{j+1}, y_{j+1} are symbols as in Fig. 9(i). Similarly, for a negative crossing, we associate four relations in Table 1(ii), where x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j and x_{j+1}, y_{j+1} are symbols as in Fig. 9(ii). For a virtual crossing, we associate four relations in Table 1(iii), where x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j and x_{j+1}, y_{j+1} are symbols as in Fig. 9(iii). For a bar, we associate two relations in Table 1(iv), where x_i, y_i and x_{i+1}, y_{i+1}, are symbols as in Fig. 9(iv). Then the set of defining relations of $\widetilde{\Pi}(D)$ are relations associated with the crossings and bars of D.

Theorem 4. *(Bourgoin [2]) The twisted knot group is an invariant of a twisted link.*

Bourgoin [2] shows that if D is a virtual link diagram, then $\widetilde{\Pi}(D)$ is the free product of the upper knot group and the lower knot group of D.

The *quandle* is a set X with a binary operator $*$, which satisfies the following.

1. For any $x \in X$, $x * x = x$.
2. For any $y \in X$, the map $S_y : X \longrightarrow X$ defined by $x \mapsto x * y$ is a bijection.
3. For any $x, y, z \in X$, $(x * y) * z = (x * z) * (y * z)$.

The quandle of a classical link was defined in [6,11]. Kauffman [10] defined the quandles $\mathbf{Q}(D)$ of a virtual link diagram D, which is a virtual link invariant.

Let $e_1, \ldots, e_p$ be the edges of D. The *twisted quandle* of D, denoted by $\widetilde{\mathbf{Q}}(D)$, is defined in a manner analogous to the twisted knot group. The generating set is $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ where x_i and y_i $(i = 1, \ldots, p)$ are symbols associated

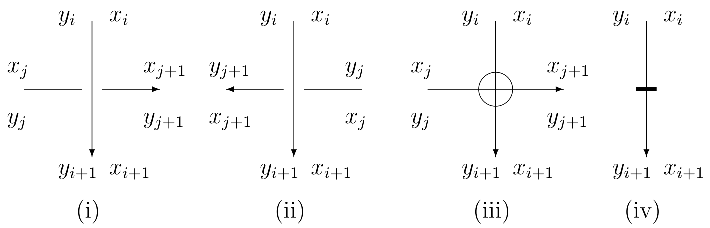


Fig. 9. Twisted knot group.

Table 1
Relations of twisted knot group.

(i)	(ii)	(iii)	(iv)
$x_{i+1} = x_i$	$x_{i+1} = x_i$	$x_{i+1} = x_i$	
$x_{j+1} = x_i^{-1} x_j x_i$	$x_{j+1} = x_i x_j x_i^{-1}$	$x_{j+1} = x_j$	$x_{i+1} = y_i$
$y_{i+1} = y_j^{-1} y_i y_j$	$y_{i+1} = y_j y_i y_j^{-1}$	$y_{i+1} = y_i$	$y_{i+1} = x_i$
$y_{j+1} = y_j$	$y_{j+1} = y_j$	$y_{j+1} = y_j$	

Table 2
Relations of twisted quandle.

(i)	(ii)	(iii)	(iv)
$x_{i+1} = x_i$	$x_{i+1} = x_i$	$x_{i+1} = x_i$	
$x_{j+1} = x_j * x_i$	$x_j = x_{j+1} * x_i$	$x_{j+1} = x_j$	$x_{i+1} = y_i$
$y_{i+1} = y_i * y_j$	$y_i = y_{i+1} * y_j$	$y_{i+1} = y_i$	$y_{i+1} = x_i$
$y_{j+1} = y_j$	$y_{j+1} = y_j$	$y_{j+1} = y_j$	

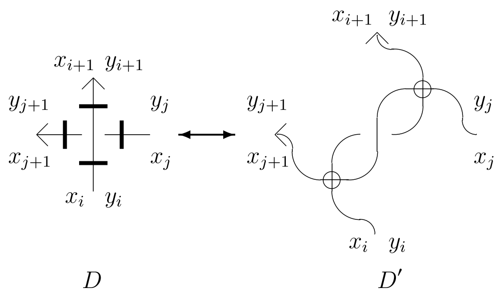


Fig. 10. Twisted knot quandles of two diagrams which are related with twisted Reidemeister move.

to each edge e_i. For a positive crossing, a negative crossing, a virtual crossing, and a bar, we associate four or two relations in Table 2(i), (ii), (iii) and (iv), respectively, where $x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j$ and x_{j+1}, y_{j+1} are symbols as in Fig. 9(i), (ii), (ii) and (iv), respectively.

Theorem 5. *The twisted quandle of a twisted link is an invariant of a twisted link.*

Proof. The proof is analogous to that of Theorem 4. We show a case that two twisted link diagrams D and D' are related with a twisted Reidemeister move of type III depicted in Fig. 10. Eight generators $x_i, y_i, x_{i+1}, y_{i+1}, x_j, y_j, x_{j+1}$, and y_{j+1} of $\widetilde{\mathbf{Q}}(D)$ (or $\widetilde{\mathbf{Q}}(D')$) are defined as in Fig. 10. For D and D', we have the same relations, $y_{i+1} = y_i$, $y_{j+1} = y_j * y_i$, $x_{i+1} = x_i * x_j$, and $x_{j+1} = x_j$. The other cases are shown similarly. □

4. Proof of Theorem 3

Let D be a twisted link diagram on S^2 with m real crossings. We construct an immersed surface $\Sigma(D)$ in $S^2 \times [-1, 1]$ associated with D. Each real crossing of D corresponds to a saddle part of a surface $\Sigma(D)$ as in Fig. 11(i). Each virtual crossing of D corresponds to two intersecting bands of $\Sigma(D)$ as in Fig. 11(ii). Each edge of D corresponds to a band of $\Sigma(D)$ as in Fig. 11(iii).

For example let D_2 and D_2' be the diagrams in Fig. 1. Then $\Sigma(D_2) = \Sigma(D_2')$, which is a surface illustrated in Fig. 12.

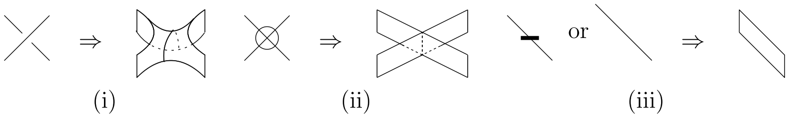


Fig. 11. A twisted link diagram and its Turaev surface.

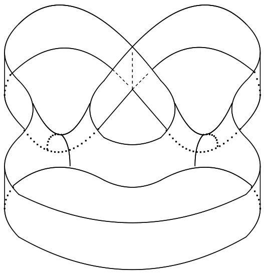


Fig. 12. An example of Turaev surface.

All loops of the boundary of $\Sigma(D)$ in $S^2 \times 1$ (or $S^2 \times -1$, resp.) correspond to loops of the state $S_A(D)$ (or $S_B(D)$) of D obtained by applying A-splice (or B-splice) at each real crossing of D. Let $\tilde{\Sigma}(D)$ be a compact surface such that $\Sigma(D)$ is regarded as the image of an immersion from $\tilde{\Sigma}(D)$ to $S^2 \times [-1, 1]$. We denote the closed surface obtained from $\tilde{\Sigma}(D)$ by attaching some disks along loops of the boundary, by $\widehat{\Sigma(D)}$. Then we see that the Euler characteristic $\chi(\widehat{\Sigma(D)})$ of $\widehat{\Sigma(D)}$ is $-m + \sharp S_A(D) + \sharp S_B(D)$.

We call the surface $\Sigma(D)$ *Turaev surface* of D. For classical cases, see [1]. Note that if the Gauss connected number is b, then $\widehat{\Sigma(D)}$ is a closed surface with b connected components and hence $\chi(\widehat{\Sigma(D)}) \leqq 2b$.

Proof of Theorem 3. Let D be a twisted link diagram with m real crossings. By definition of $\langle\langle D|S\rangle\rangle$, we have $\operatorname{Max}\deg_A \langle\langle D|S_A(D)\rangle\rangle = m + 2\sharp S_A(D)$ and $\operatorname{Min}\deg_A \langle\langle D|S_B(D)\rangle\rangle = -m - 2\sharp S_B(D)$. So we see that

$$\begin{aligned}\operatorname{Max}\deg_A \langle\!\langle D \mid S_A(D)\rangle\!\rangle - \operatorname{Min}\deg_A \langle\!\langle D \mid S_B(D)\rangle\!\rangle &= 2m + 2\big(\sharp S_A(D) + \sharp S_B(D)\big)\\ &= 2m + 2\big(m + \chi\big(\widehat{\Sigma(D)}\big)\big)\\ &\leqq 4(m+b).\end{aligned}$$

Let S be a state of D obtained from D obtained by applying B-splices (or A-splices) at k real crossings and A-splices (or B-splices) at the others. Thus we have $\sharp S_A - k \leqq \sharp S \leqq \sharp S_A + k$ (or $\sharp S_B - k \leqq \sharp S \leqq \sharp S_B + k$). Then we see that

$$\begin{aligned}&\operatorname{Max}\deg_A \langle\langle D|S\rangle\rangle = m - 2k + 2\sharp S \leqq \operatorname{Max}\deg_A \langle\!\langle D \mid S_A(D)\rangle\!\rangle\\ &\big(\text{or } \operatorname{Min}\deg_A \langle\langle D \mid S\rangle\rangle = -m + 2k - 2\sharp S \geqq \operatorname{Min}\deg_A \langle\!\langle D \mid S_B(D)\rangle\!\rangle\big)\end{aligned}$$

for a state S of D, which means that $\operatorname{Max}\deg_A \langle\langle D\rangle\rangle \leqq \operatorname{Max}\deg_A \langle\langle D|S_A(D)\rangle\rangle$ (or $\operatorname{Min}\deg_A \langle\langle D\rangle\rangle \geqq \operatorname{Min}\deg_A \langle\langle D|S_B(D)\rangle\rangle$). Therefore we have the result. □

5. Applications

For $n \in \mathbb{N}$, let X_n be a dihedral quandle, which is $\mathbb{Z}/n\mathbb{Z}$ with $x * y = 2y - x$. For a twisted link diagram D, let $e_1, \dots, e_p$ be the edges of D and $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ the generating set of the quandle $\widetilde{\mathbf{Q}}(D)$ as before. A *coloring* of D by X_n is a homomorphism from $\widetilde{\mathbf{Q}}(D)$ to X_n, which is a map sending the generators to elements of X_n (denoted by the same symbols) such that they satisfy the equations in Table 2(i), (ii), (iii) and (iv). The number of colorings of D is denoted by $\widetilde{\mathrm{col}}_n(D)$.

Proposition 6. *$\widetilde{\mathrm{col}}_n(D)$ is an invariant of a twisted link D.*

Proposition 7. *(cf. [9]) Let D be a twisted link diagram. If $\widetilde{\mathrm{col}}_n(D)$ is less than $n \times n$, D is not equivalent to any virtual link.*

Proof. If D is equivalent to a virtual link, there is a virtual link diagram D' which is equivalent to D. Then the quandle of D is a free product of upper quandle and lower quandle. Thus $\widetilde{\mathrm{col}}_n(D')$ is greater than or equal to $n \times n$ since the number of upper (or lower) dihedral quandle of order n is greater than or equal to n. □

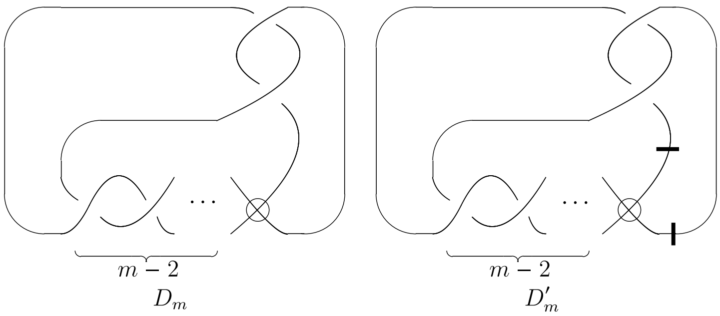


Fig. 13. Twisted links whose twisted Jones polynomial is the same as virtual knot.

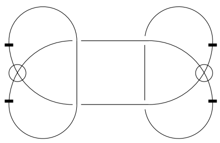


Fig. 14. A twisted link diagram of the trivial knot.

For two twisted link diagrams D_2 and D'_2 in Fig. 1(i) and (ii), we have $\widetilde{\mathrm{col}}_3(D_2) = 9$ (or $\widetilde{\mathrm{col}}_3(D'_2) = 3$) in Fig. 1(i) (or (ii)). Thus the twisted link diagrams in Fig. 1(i) and (ii) are not equivalent. The diagram in Fig. 1(ii) is not equivalent to any virtual link diagram since $\widetilde{\mathrm{col}}_3(D'_2) = 3$ by Proposition 7. They are also distinguished by the twisted link groups [2].

For $m > 1$, the twisted Jones polynomials of two twisted links presented by the diagrams D_m and D'_m in Fig. 13 are $-A^{-6}(A^4 + A^{-4}) - A^{-4m}(A^3 - A^{-3})(A + A^{-1})$ (or $-A^6(A^4 + A^{-4}) + A^{-4m+12}(A^3 - A^{-3})(A + A^{-1})$) if m is even (or odd).

Our invariant of D_m is $-A^{-6}(A^4 + A^{-4}) - A^{-4m}(A^2 - A^{-2})((A^2 + A^{-2})d_1 + 1)$ (or $-A^6(A^4 + A^{-4}) + A^{-4m+12}(A^2 - A^{-2})((A^2 + A^{-2})d_1 + 1)$) if m is even (or odd). Our invariant of D'_m is the same as the twisted Jones polynomial of D_m. Thus the twisted link diagrams D_{m_1} and D'_{m_2} are not equivalent for any $m_1, m_2 > 1$.

The real crossing number of the twisted link presented by D_m (or D'_m) is m from Theorem 3 since $\mathrm{Span}_A(R_{D_m}) = 4m+2$ (or $\mathrm{Span}_A(R_{D'_m}) = 4m+2$).

Thus D_{m_1} is not equivalent to D_{m_2} and D'_{m_1} is not equivalent to D'_{m_2} for $m_1 \neq m_2$.

We have $\widetilde{\mathrm{col}}_n(D_m) = n \times n$. We obtain $\widetilde{\mathrm{col}}_n(D'_m) = n$ if n and $4m-6$ are coprime, which implies that the twisted link diagrams D_m and D'_m are not equivalent. Furthermore the diagram D'_m is not equivalent to any virtual link diagram by Proposition 7.

Therefore we have the following theorem.

Theorem 8. *For $m > 1$, there is a twisted knot such that*

(1) *its real crossing number is m,*
(2) *it is not a virtual knot, and*
(3) *its twisted Jones polynomial is the same as a virtual knot.*

Theorem 9. *The twisted Reidemeister move of type III in Fig.* 4 *is not a consequence of the other moves in Figs.* 2, 3 *and* 4.

Let η be the map from {twisted link diagrams} to {virtual link diagrams} such that for a twisted link diagram, $\eta(D)$ is the virtual link diagram obtained from D by forgetting all bars of D.

Proof of Theorem 9. If a twisted link diagram D is related to a twisted link diagram D' with one of Reidemeister moves, virtual Reidemeister or twisted Reidemeister moves except the twisted Reidemeister move of type III, then the polynomial invariants are equal: $R_{\eta(D)} = R_{\eta(D')}$. The following example shows that there is a pair of twisted link diagrams D and D' such that D' is obtained from D by applying the moves including the twisted Reidemeister move of Type III in Fig. 4 and that $R_{\eta(D)} \neq R_{\eta(D')}$. $\square$

Example. The twisted link diagram D in Fig. 14 is related to a trivial link diagram with the sequence of Reidemeister moves, virtual Reidemeister or twisted Reidemeister moves including twisted Reidemeister of type III as depicted in Fig. 15. The virtual knot $\eta(D)$ is Kishino's knot, which means that $R_{\eta(D)} \neq 1$ [4,12]. On the other hand, $R_{\eta(D')} = 1$.

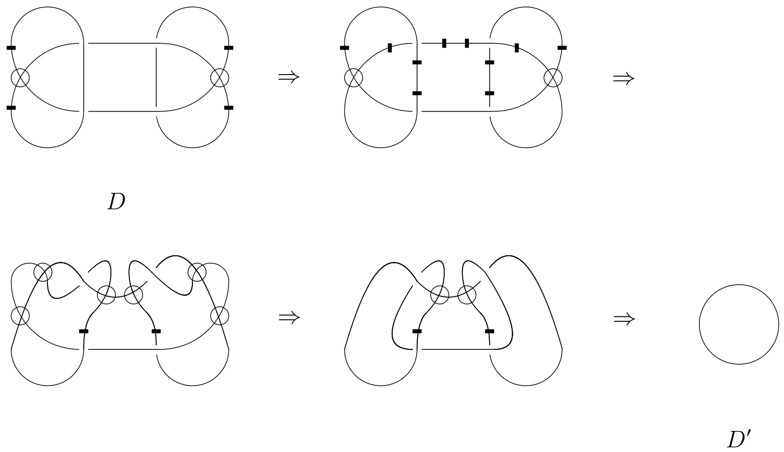


Fig. 15. Moves to a trivial diagram.

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